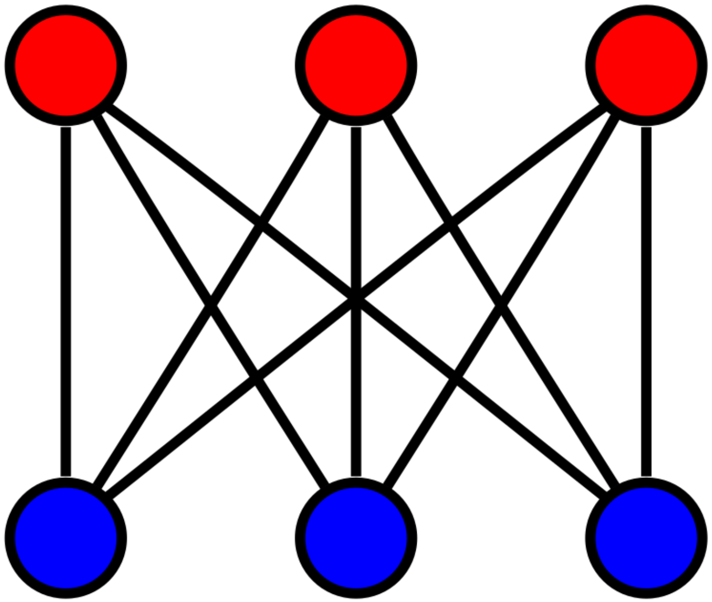
Graph Theory

Planarity

PLANARITY



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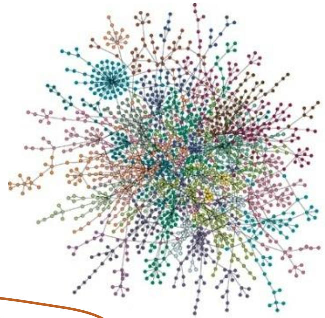

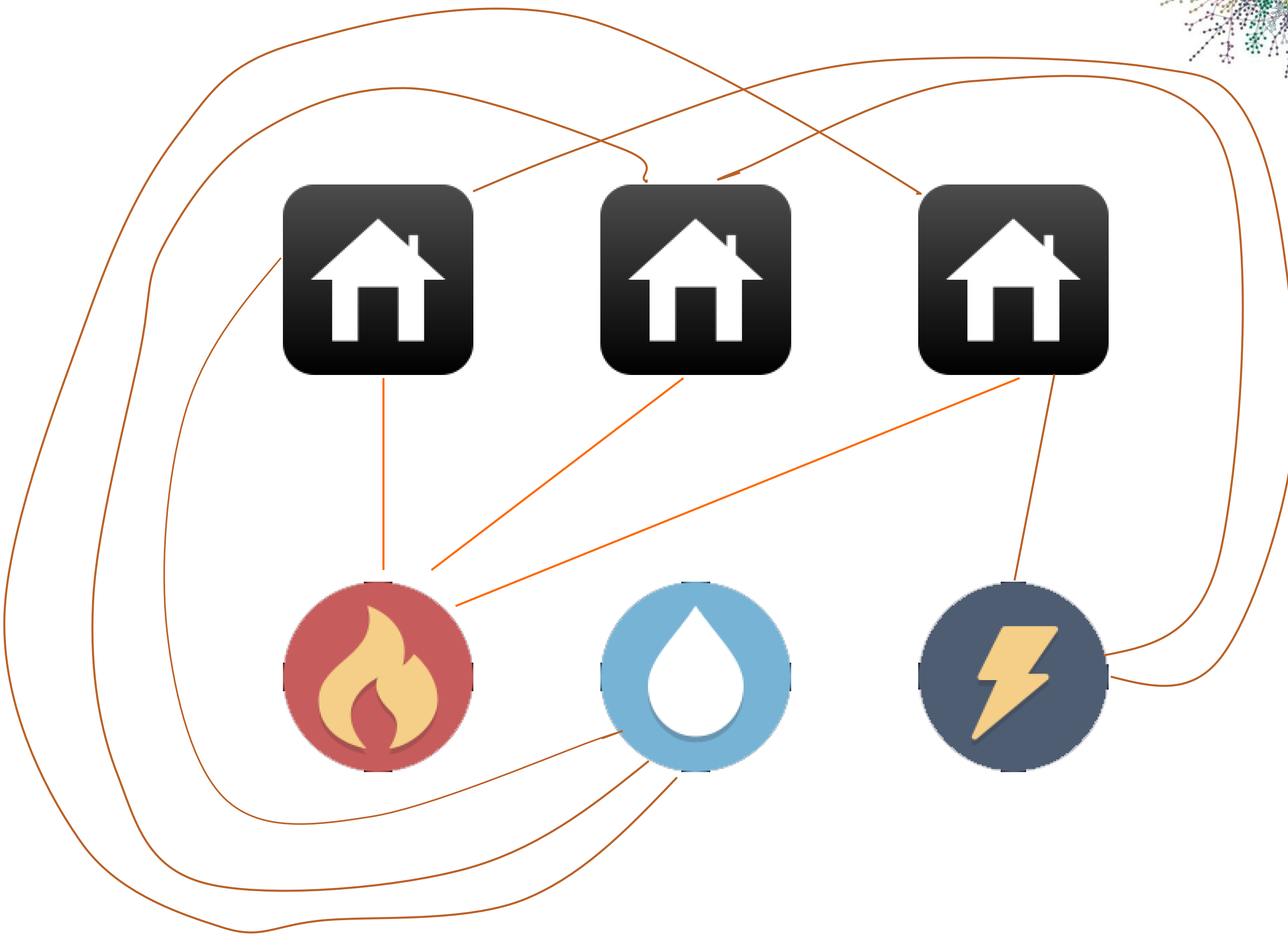

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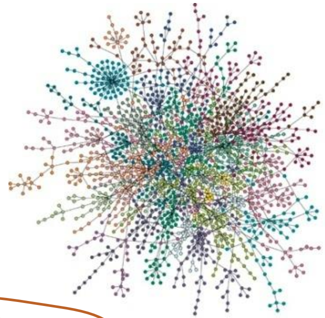

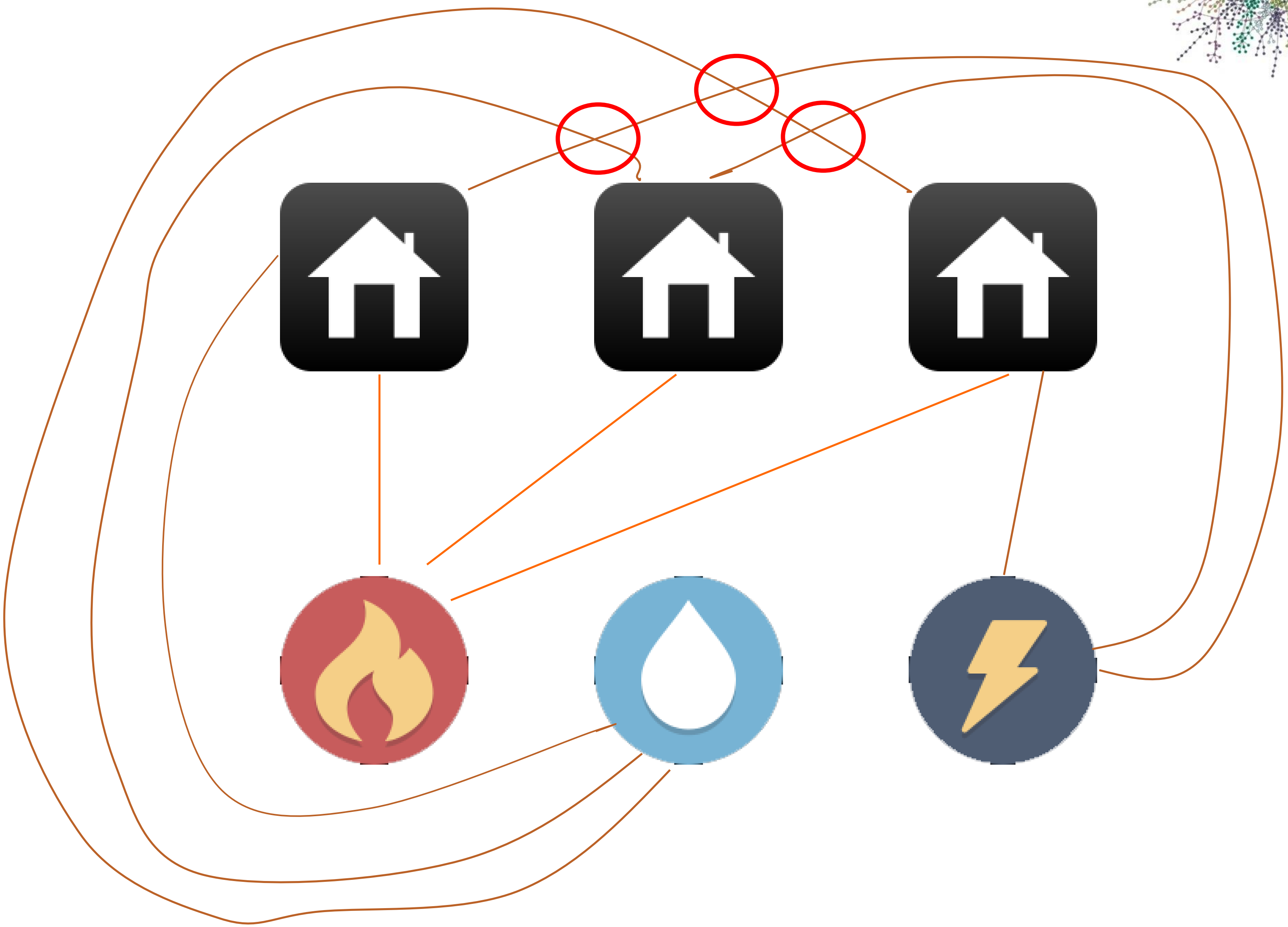

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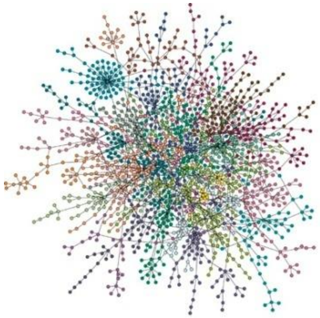
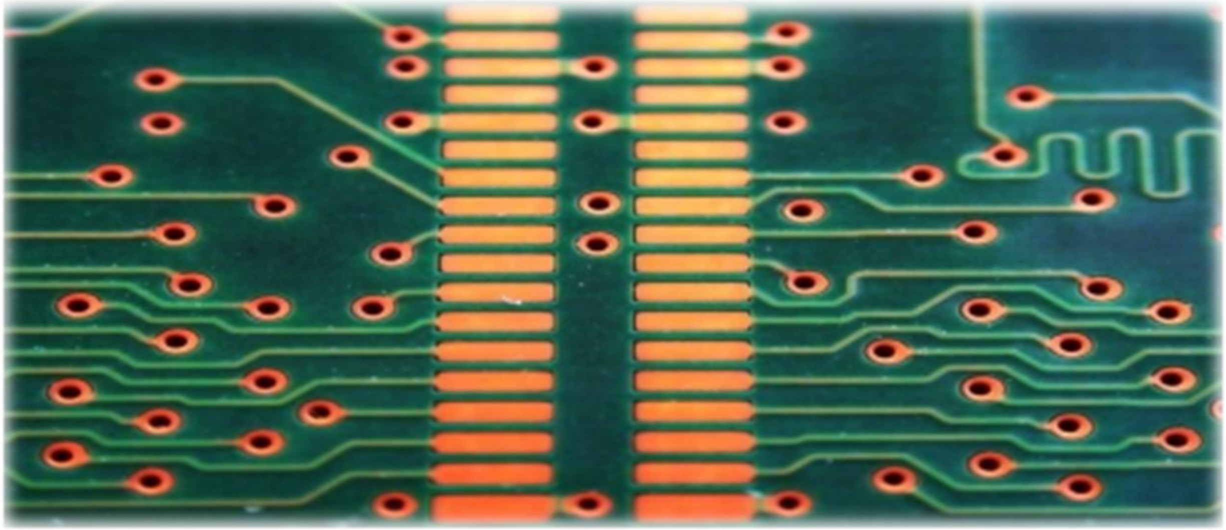
- **The Problem:**
  - Can we construct a graph in which there do not exist intersecting edges???





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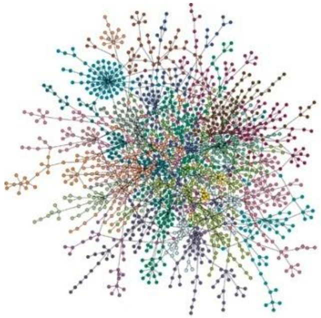
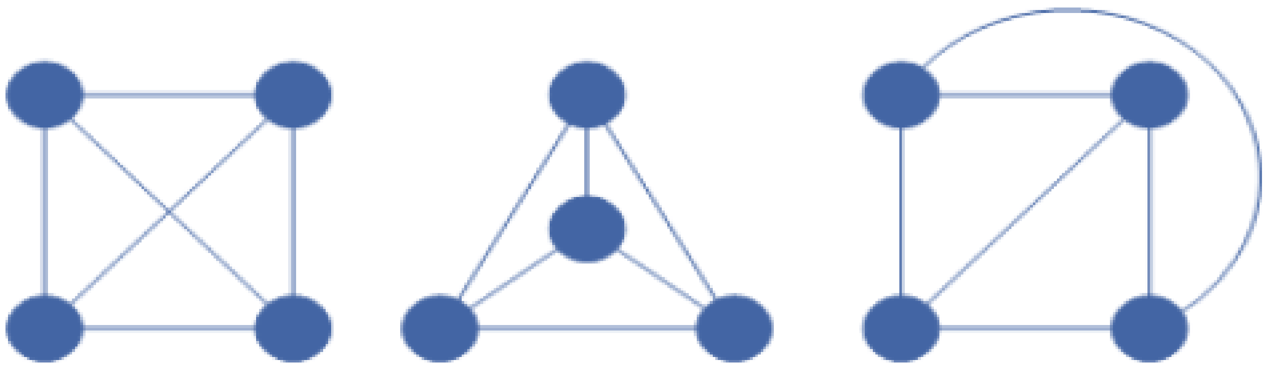


Planar Graphs

- A **plane graph** is a graph that any two of its edges are only adjunct on their endpoints.
- A graph is called **planar** or **embeddable in the plane**, if is isomorphic to a plane graph.



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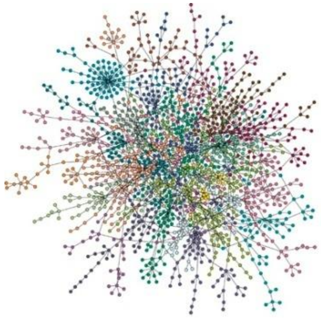
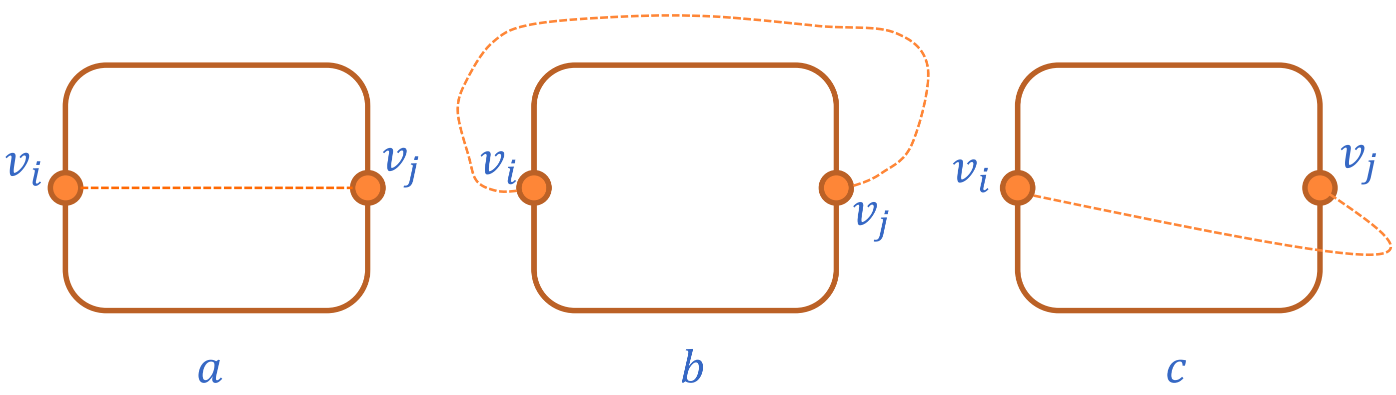


Planar Graphs

- **Jordan Curve**: A continuous line in the plane that does not intersect itself.
- **Closed Jordan Curve**: A Jordan curve whose two ends coincide.
- **Theorem 1 (Jordan):**

  Given a closed Jordan curve L and its two points v_i and v_j, then the Jordan curve joining these points,

  a) either is inside L,
  b) either outside L,
  c) or intersects L in some points other than v_i and v_j.



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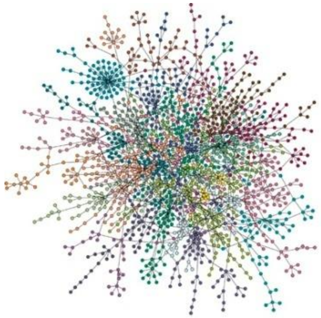
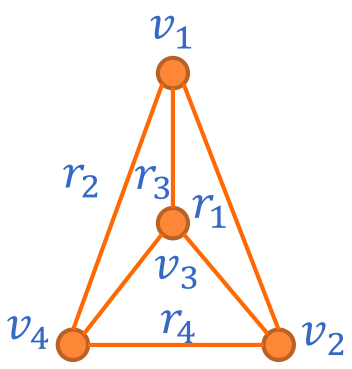


Planar Graphs

- Given a planar graph G and a point x of the layer, we call **region** or **face** or the **window** of G containing the x, the set of points of the plane that can be joined to x through a Jordan curve that does not intersects the edges of G.
- **r (or f)** denotes the number of regions of a planar graph
- The **boundary** of a region is the subgraph affected by the edges and vertices adjacent to the region (i.e., the edges surrounding the region).



Boundary		
Region	**Edges**	**Vertices**
r_1	$(v_1, v_2), (v_2, v_3), (v_3, v_1)$	v_1, v_2, v_3
r_2	$(v_1, v_2), (v_2, v_4), (v_4, v_1)$	v_1, v_2, v_4
r_3	$(v_1, v_3), (v_3, v_4), (v_4, v_1)$	v_1, v_3, v_4
r_4	$(v_2, v_3), (v_3, v_4), (v_4, v_2)$	v_2, v_3, v_4

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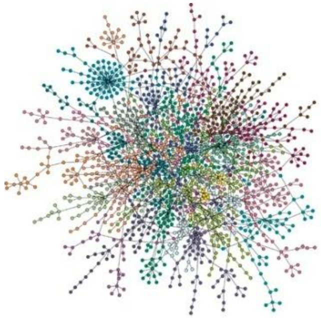

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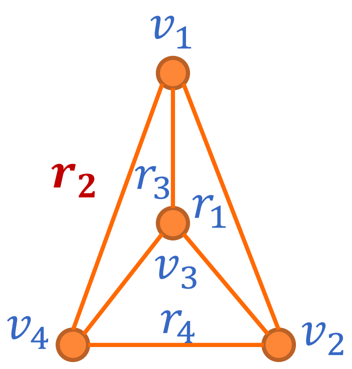

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r_3	$(v_1, v_3), (v_3, v_4), (v_4, v_1)$	v_1, v_3, v_4
r_4	$(v_2, v_3), (v_3, v_4), (v_4, v_2)$	v_2, v_3, v_4

- The region $\boldsymbol{r_2}$ is called, **exterior** or **infinite** or **unbounded** or **outer**.

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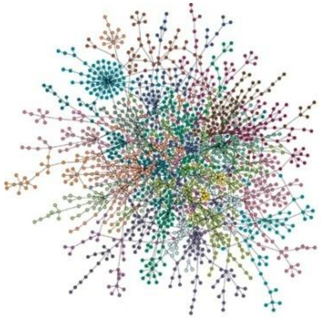

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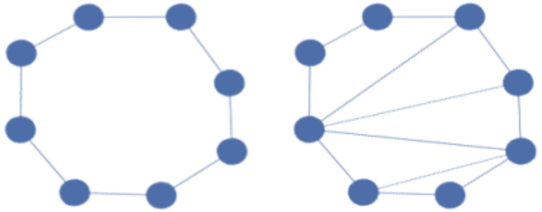
- **r (or f)** denotes the number of regions of a planar graph
- The **boundary** of a region is the subgraph affected by the edges and vertices adjacent to the region (i.e., the edges surrounding the region).
- **Outer-planar** is called a graph if all of its vertices belong in one region.
  - The edges of such a graph lie either on top or in a circle and they don't intersect.
  - Each outer-planar graph is planar but the reverse is not true (eg K_4 is planar but not outer-planar).



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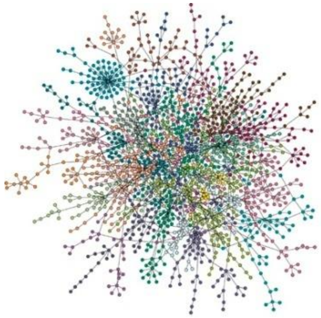

Euler & Kuratowski Theorems

- **Euler's Theorem** proves that despite the way we embed a graph in the plane, the number of regions remains constant and is given by the **Euler's polyhedron formula**.

- **Theorem 2 (Euler, 1752):**

If G is a connected plane graph, then it holds that:

$$n + r = m + 2$$

$\lvert V\rvert$	$\lvert E\rvert$	r	$R - \lvert E\rvert + \lvert V\rvert$
1	0	1	2

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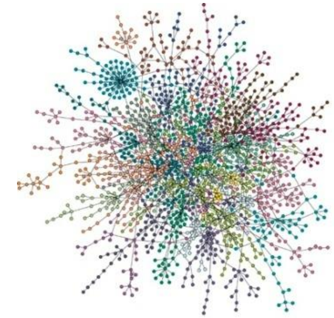

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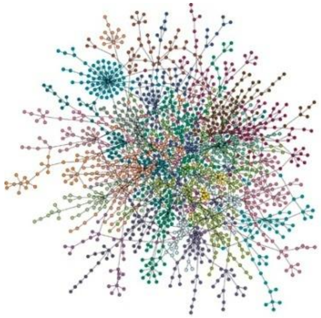

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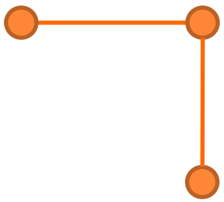

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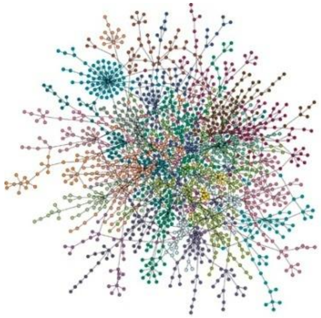

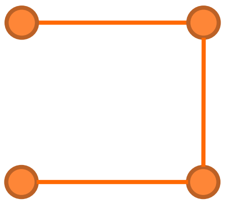

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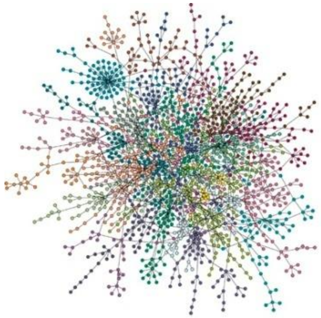

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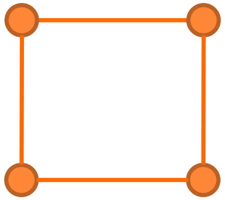

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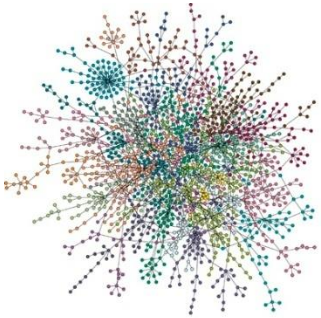

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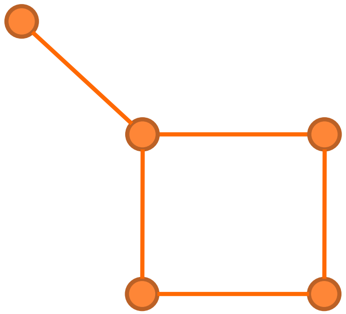

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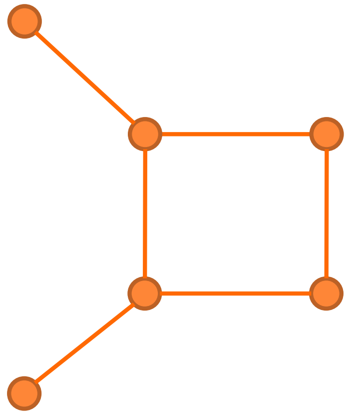

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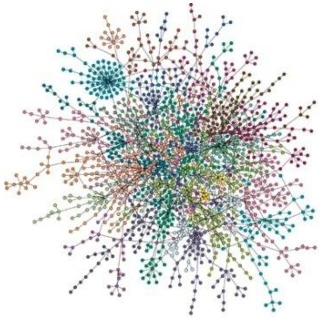

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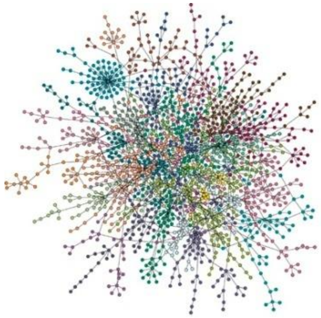

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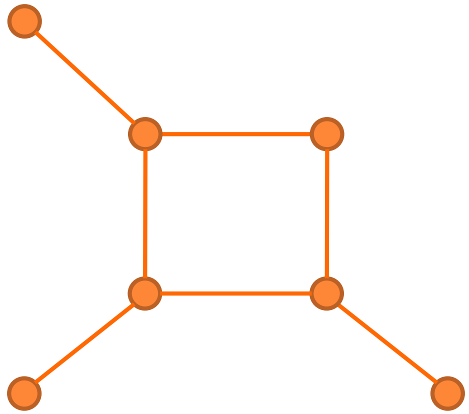

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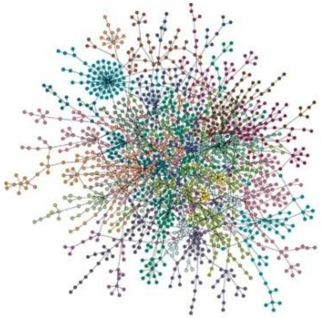

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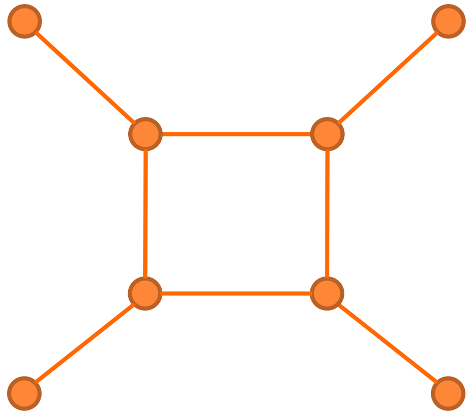

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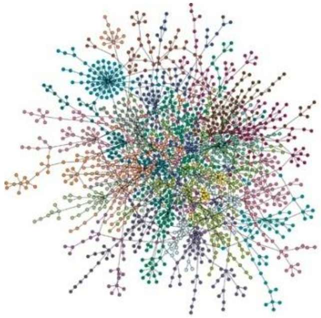

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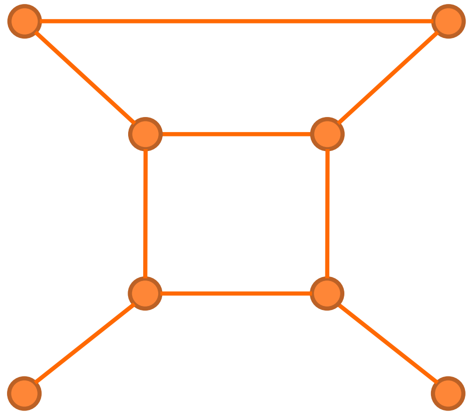

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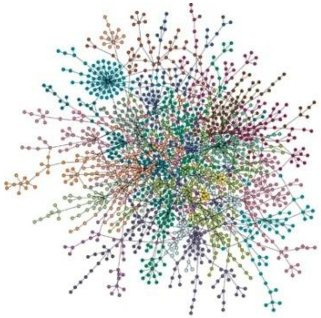

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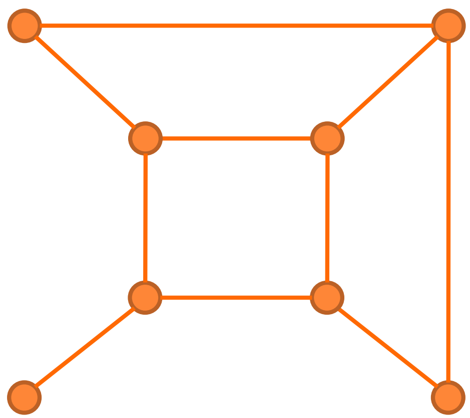

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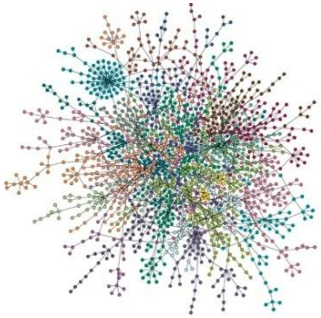

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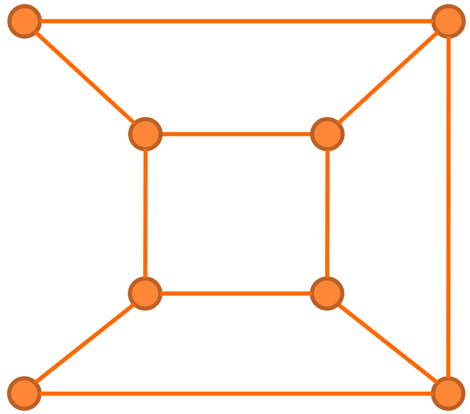

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8	11	5	2

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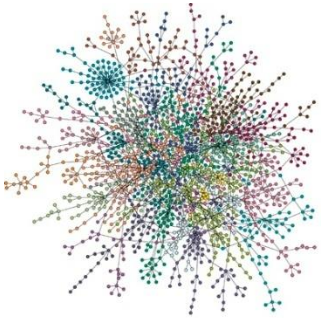

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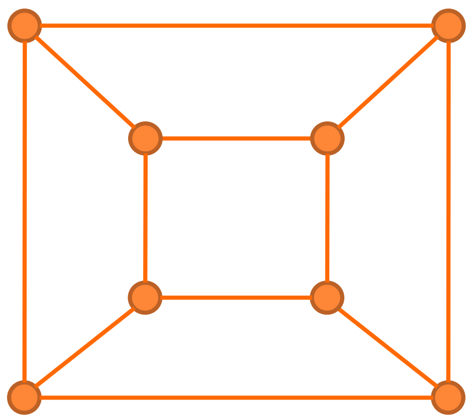

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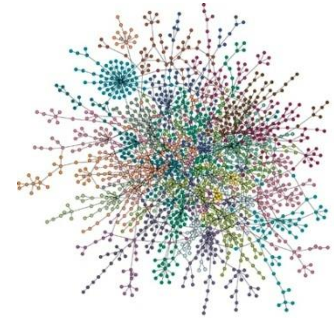

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Inductively on the number of edges…

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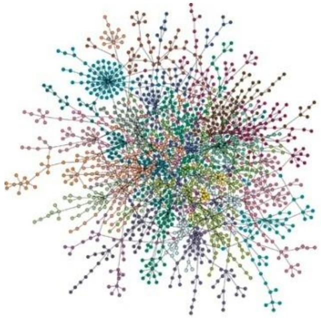

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- **Euler's Theorem** proves that despite the way we embed a graph in the plane, the number of regions remains constant and is given by the **Euler's polyhedron formula**.

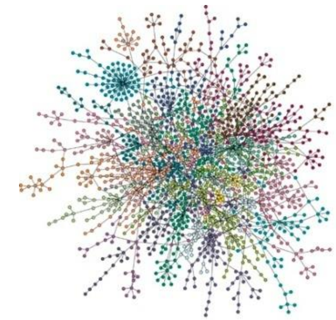
- **Theorem 2 (Euler, 1752):**

  If G is a connected plane graph, then it holds that:

$$n + r = m + 2$$

  - If $m = 0$, then $n = 1$ (as G is connected), and $r = 1$.
  - Let that the Theorem holds for a connected graph of $m - 1$ edges.
  - In this graph we draw a new edge e, and three cases may occur:

PLANARITY



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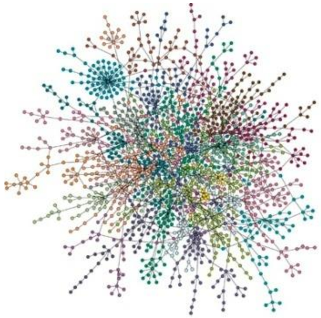
    1. The new edge e is a loop and hence a new region is formed, as the number of vertices remains constant.

Planarity



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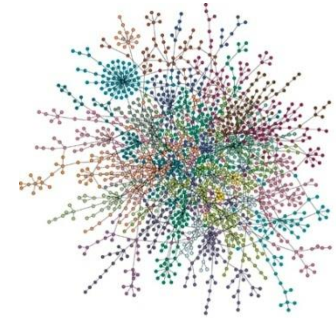
2. The new edge e connects two existing vertices, and hence, a new region is formed on a constant number of edges, and

Planarity



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      3. The new edge e is incident on only one vertex of G, and hence a new vertex is constructed as the number of regions remains the same.

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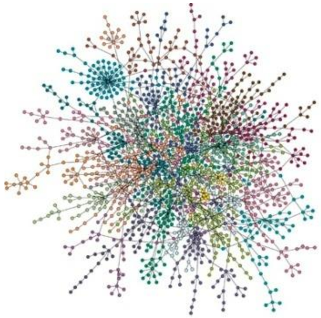

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  - In this graph we draw a new edge e, and three cases may occur:
    - In any case, the Theorem is correct.

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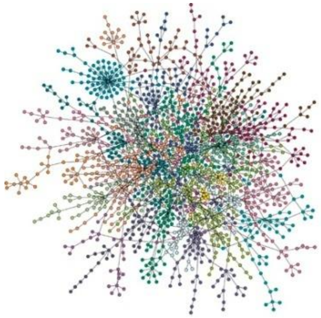

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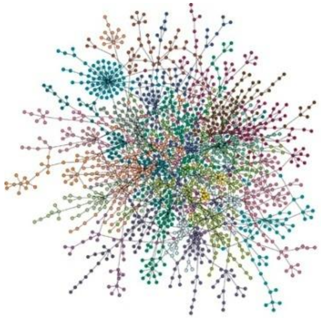

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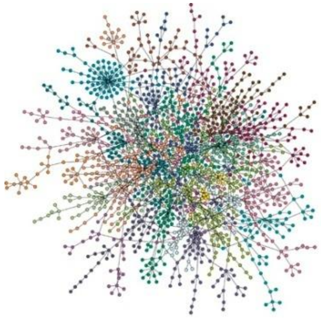

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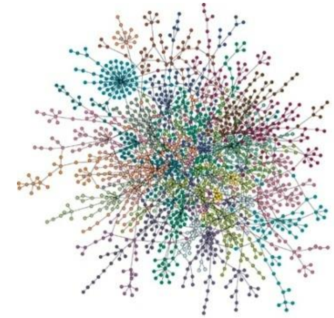

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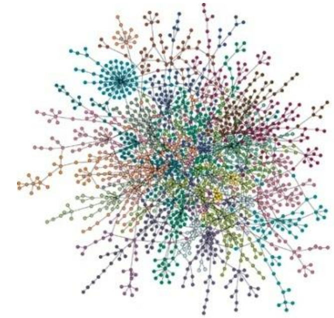
- $n_1 + r_1 = m_1 + 2$
$$n_2 + r_2 = m_2 + 2$$
$$\dots$$
$$n_k + r_k = m_k + 2$$
- $\Longrightarrow n + r + (k - 1) = m + 2k$

Planarity



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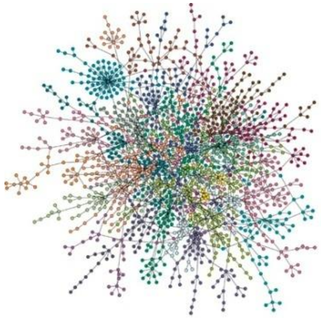

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    $\boldsymbol{r = r_1 + r_2 + \cdots + r_k - (k - 1)}$
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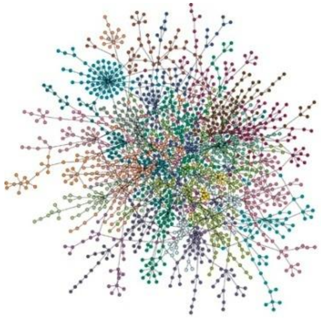

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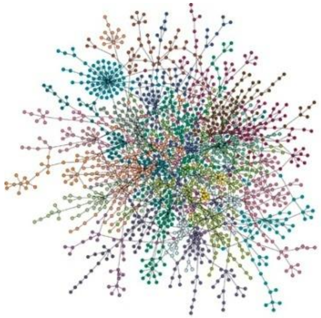

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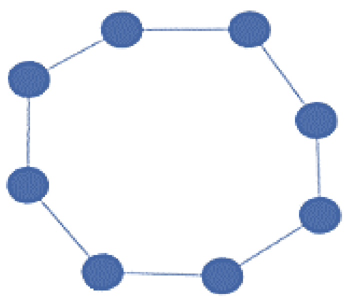

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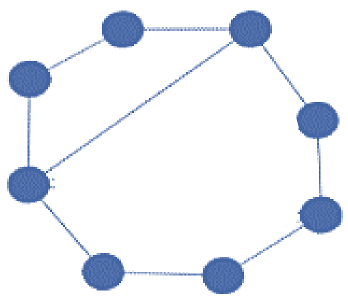

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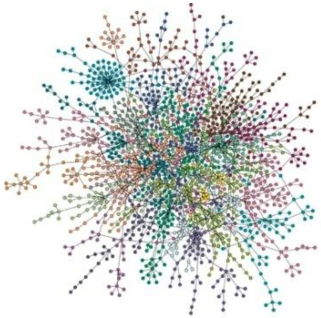

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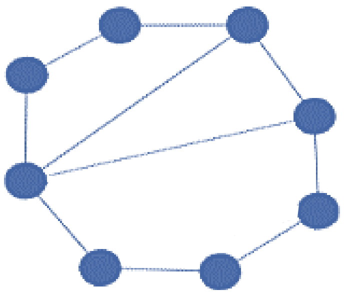

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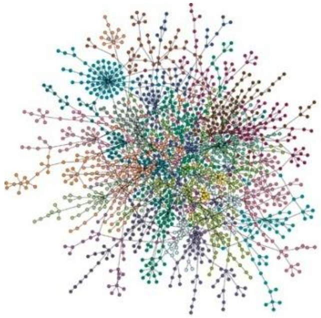

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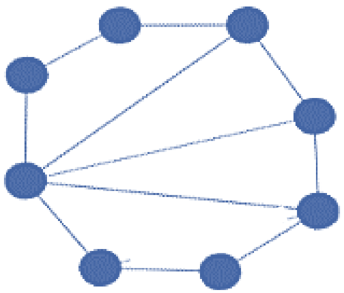

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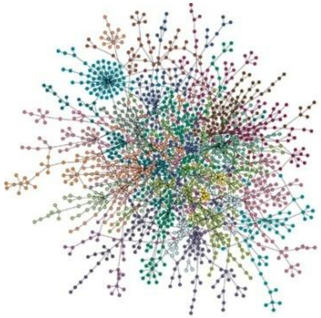

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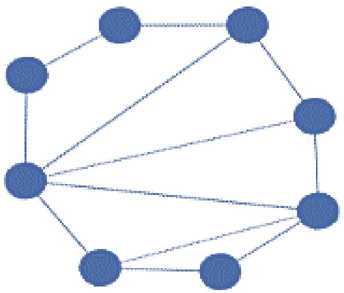

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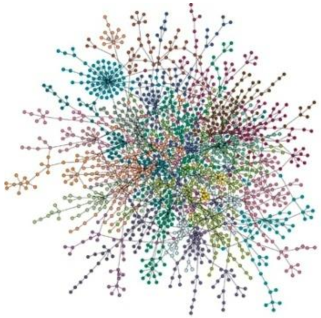

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    Such a process may proceed until a "specific point"…
    - **Maximal plane graph**, is called the graph G if for each pair x, y of discrete non adjacent vertices of G, the graph $G + (x, y)$ is not plane. *But … until where …*
    - If there exist region surrounded by cycle of length 4, then a new edge may be added and the graph remain plane.
    - As long as there exist regions surrounded by cycles of length greater than 3 there can be added edges retaining the planarity of the graph.

      Therefore, the maximal plane graphs are called **triangulated**.

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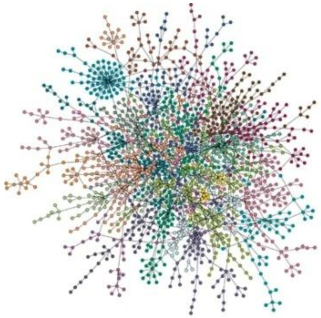

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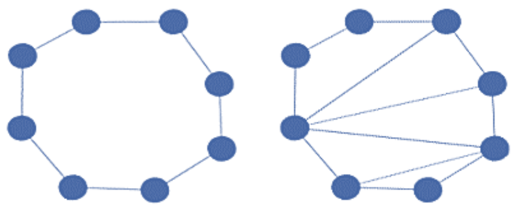
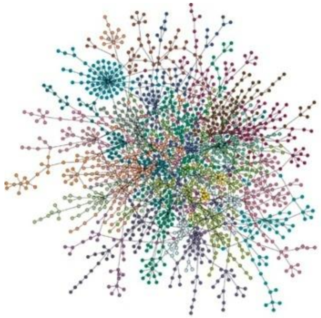
    Such a process may proceed until a "specific point"…

    - **Maximal plane graph**, is called the graph G if for each pair x, y of discrete non adjacent vertices of G, the graph $G + (x, y)$ is not plane.
    - Similarly are defined the **Maximal outer planar graphs**, that are produced after the triangulation of a polygon, while every maximal plane graph occurs after the triangulation of the sphere.



Planarity



Euler & Kuratowski Theorems

- **Euler's Theorem** proves that despite the way we embed a graph in the plane, the number of regions remains constant and is given by the **Euler's polyhedron formula**.

- **Lemma (Handshake Lemma for Plane Graphs)**:

  For each connected plane graph G it holds that:

$$2m = \sum_{i=1}^{r=m-n+2} d(r_i) = \sum_{j=D(G)}^{D(G)} j\, n(j)$$

  where $d(r_i)$ is the degree of the region r_i, i.e. the number of the edges surrounding the i-th region, while $n(j)$ denotes the number of vertices of degree j.

  - Each edges counts twice in each regions and each vertex.

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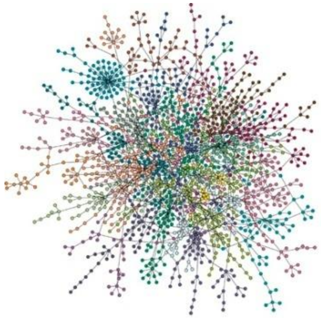

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- **Euler's Theorem** proves that despite the way we embed a graph in the plane, the number of regions remains constant and is given by the **Euler's polyhedron formula**.

- **Corollary** :

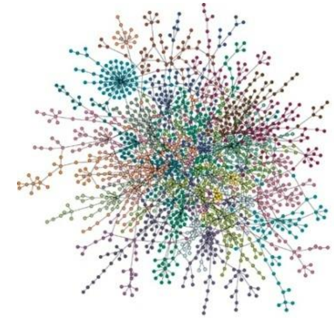
  If G is a connected maximum planar graph of $n \geq 3$ edges it holds that:

$$m = 3n - 6$$

  - Let r be the number of areas of the graph.
  - At a maximum plane graph there is: $d\,(r_i) = 3$ for each area.
  - The Lemma therefore states:

$$2m = 3 + 3 + \cdots + 3(r = m - n + 2\ times) \Rightarrow 2m = 3(m - n + 2) \Rightarrow m = 3n - 6$$

Planarity



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- **Euler's Theorem** proves that despite the way we embed a graph in the plane, the number of regions remains constant and is given by the **Euler's polyhedron formula**.
- **Corollary** :

  For every connected planar graph with $n \geq 3$ vertices it holds that:

  $$m \leq 3n - 6$$

  - Let r be the number of areas of the graph.
  - In a simple plane graph, $d(r_i) \geq 3$ applies to each area r_i.
  - The Lemma therefore states:

  $$2m \geq 3 + 3 + \cdots + 3\ (r = m - n + 2\ times) \Rightarrow 2m \geq 3\ (m - n + 2) \Rightarrow m \leq 3n - 6$$

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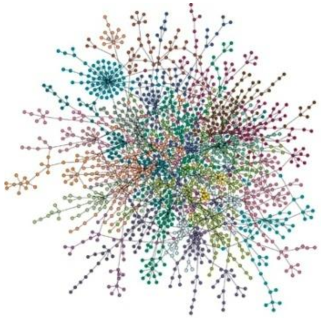

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- **Euler's Theorem** proves that despite the way we embed a graph in the plane, the number of regions remains constant and is given by the **Euler's polyhedron formula**.
- **Corollary** :

  For every connected plane bipartite graph G with $n \geq 3$, it holds that:

  $$m \leq 2n - 4$$

  - Let r be the number of areas of the graph.
  - On a plane bipartite it holds that, $d(r_i) \geq 4$ for each region r_i.
  - The Lemma therefore states:

  $2m \geq 4 + 4 + \cdots + 4 \ (r = m - n + 2 \ times) \Rightarrow 2m \geq 4 \ (m - n + 2) \Rightarrow m \leq 2n - 4$

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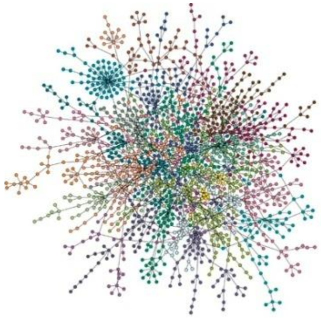

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- **Corollary** :

  Each plane graph contains at least one vertex v of degree $d(v) \leq 5$

  - Suppose that all vertices have degree ≥ 6 and that the graph has n vertices and m edges.
  - It holds that $m \leq 3n - 6$ or $2m \leq 6n - 12$ [1]
  - From the Handshake Lemma we know that the sum the degrees of the vertices of a graph are $2m$.
  - Since $d(v) \geq 6$ for every v it holds $2m \geq 6n$ [2]
  - From [1] and [2] $\Rightarrow 6n \leq 2m \leq 6n - 12$, that is a contradiction.

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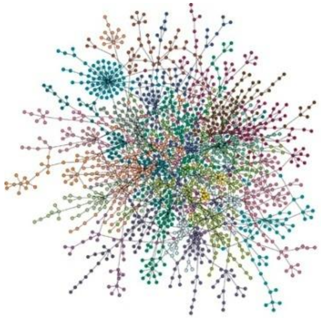

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- **Euler's Theorem** proves that despite the way we embed a graph in the plane, the number of regions remains constant and is given by the **Euler's polyhedron formula**.

- **Theorem 3 :**

  The graph K_5 is non plane.
  - If K_5 was plane, then from the Corollary it would hold that $m \leq 3n - 6$
  - Now it holds that $1 = |E(K_5)| \leq\ 3(5) - 6 = 9$, that is a contradiction.

- **Theorem 4 :**

  The graph $K_{3,3}$ is non plane.
  - Let that $K_{3,3}$ is plane graph…
  - Since the graph does not contain triangular regions it is implied that every regions surrounded by polygons of at least 4 vertices.
  - Then, it holds that $4r \leq 2m = 18$, but it should hold $r \leq 4$.
  - Euler's formula→ $2 = n - m + r \leq 6 - 9 + 4 = 1$, that is a contradiction

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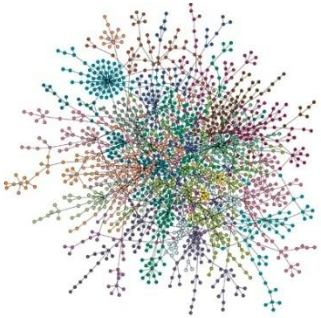


Euler & Kuratowski Theorems

- K_5 is the non-plane graph with the smallest number of vertices and $K_{3.3}$ the non-plane graph with the smallest number of edges.
- Two graphs are called **homomorphic** if one can occur from the other with one or more subdivisions of its edges.
- **Theorem 5 (Kuratowski 1930):**

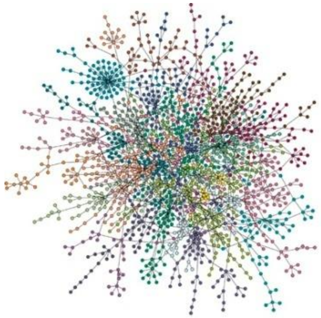
  A graph is plane if it does not contain subgraphs homomorphic to K_5 and $K_{3.3}$.
- **Theorem 6 (Wagner 1937, Harray & Tutte 1965):**

  A graph is plane if it does not contain subgraphs contractible to K_5 and $K_{3.3}$.
  - **contraction** is its reverse procedure of edge subdivision
- A graph is embeddable on the surface of a sphere, iff it is embeddable in the plane.

Planarity

- **Embedding Graphs to Multiple Layers**
  - Which is the minimum number of levels required for the embed of a graph?

Planarity



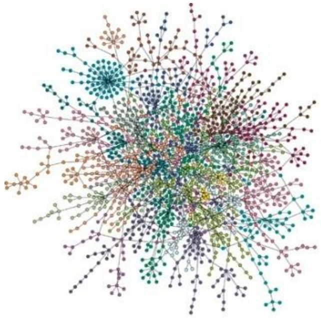
Embedding Graphs to Multiple Layers

- Which is the minimum number of levels required for the embed of a graph?
  - **Thickness $t(G)$**, defines the minimum number of levels required for the integration of a graph ("how much non-planar is a graph?").
    - A graph G is decomposed to $r > 2$ planar graphs: $G = H_1 \cup \mathrm{H}_2 \cup \cdots \cup \mathrm{H}_r$
    - The thickness of a plane graph is $t = 1$
    - $t(K_{3,3}) = t(K_5) = t(K_8) = 2$
  - **Corollary:**

The thickness of a connected graph $G, n \geq 3$ satisfies the equations:

$$t(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil = \left\lfloor \frac{m+3n-7}{3n-6} \right\rfloor$$

PLANARITY



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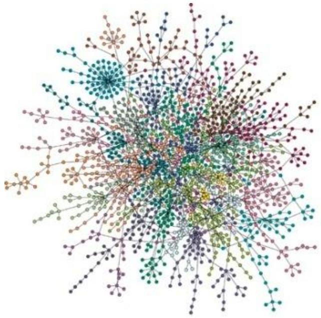
    - A graph G is decomposed to $r > 2$ planar graphs: $G = H_1 \cup \mathrm{H}_2 \cup \cdots \cup \mathrm{H}_r$
    - The thickness of a plane graph is $t = 1$
    - $t(K_{3,3}) = t(K_5) = t(K_8) = 2$
  - **Corollary:**

    The thickness of a bipartite graph G of n vertices and m edges satisfies the equation:

$$t(G) \geq \left\lceil \frac{m}{2n-4} \right\rceil$$

PLANARITY



Embedding Graphs to Multiple Layers

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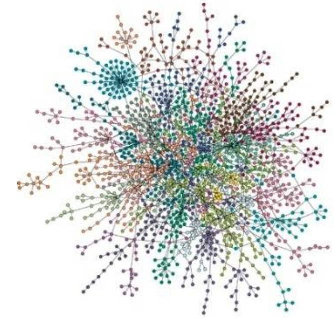
    The thickness of a complete graph K_n, of $n \geq 3$ vertices satisfies the equation:

$$t(K_n) \geq \left\lceil \frac{n+7}{6} \right\rceil$$

Planarity



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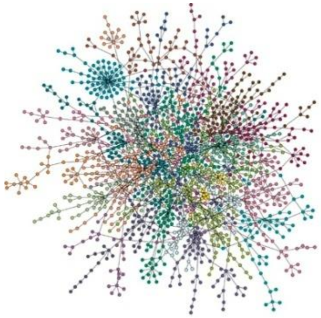
$$t(K_n) = \begin{cases} \left\lfloor \frac{n+7}{6} \right\rfloor & if\ n \neq 9{,}10 \\ 3 & if\ n = 9{,}10 \end{cases}$$

Planarity



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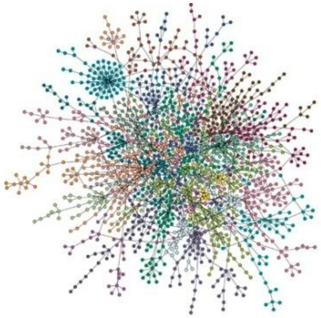
    The thickness of a complete bipartite graph $K_{m,n}$, satisfies the equation:

$$t(K_{m,n}) \geq \left\lceil \frac{mn}{2(m+n-2)} \right\rceil$$

Planarity



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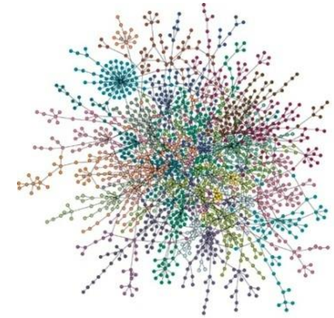
- Which is the minimum number of edge sections of a non-planar graph?
  - **Crossing number $cr(G)$**, defines the minimum number of sections of a graph per plane.
    - Crossing number of a plane graph is $cr = 0$
    - $cr(K_{3,3}) = cr(K_5) = 1$
  - **Theorem 8:**
    For the crossing number of the complete connected graph K_6, it holds:
$$cr(K_6) = 3$$

PLANARITY



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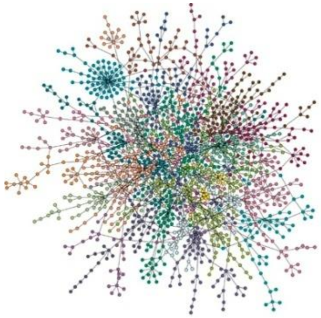
  - **Theorem 9:**
    The crossing number of the complete connected graph K_n, and the complete bipartite graph K_{n_1,n_2} satisfy the equations:

$$cr(K_n) \leq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

Planarity

Embedding Graphs to Multiple Layers



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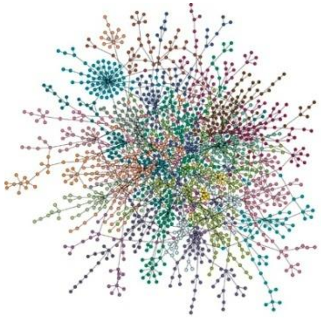
PLANARITY



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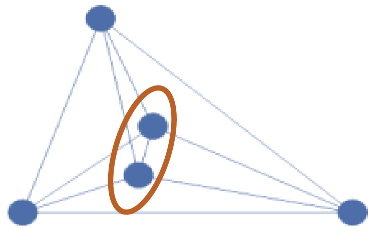

PLANARITY

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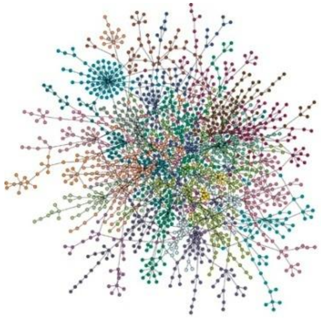
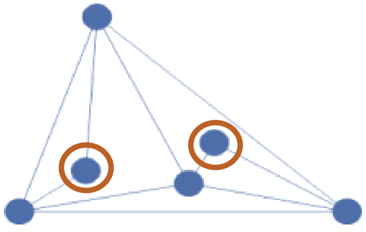
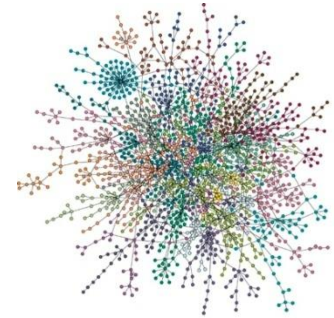




Planarity



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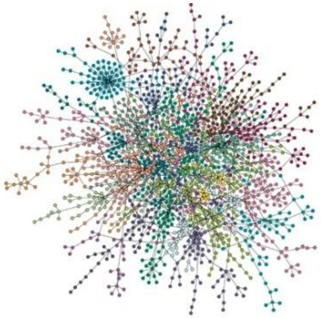
    - **Theorem 10:**
      The split number of K_n, K_{n_1,n_2} satisfy the equations:

$$s(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil \qquad , n \geq 10$$

PLANARITY



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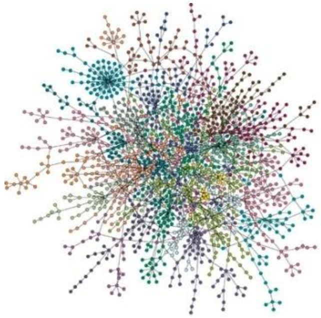
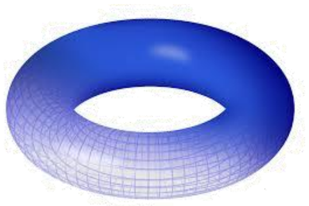
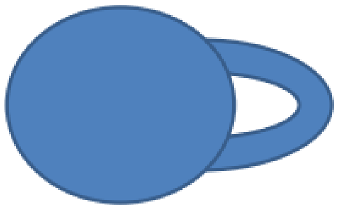
PLANARITY



- **Embedding Graphs to Multiple Layers**
  - Embed to other surfaces …
    - Embed into a “torus”.

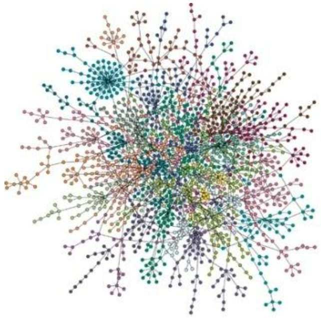


    - A “torus” is homomorphic to a “handle”.



    - Moebius Band.

Planarity



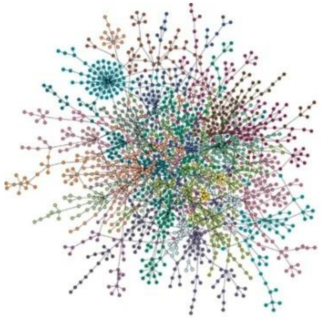
Embedding Graphs to Multiple Layers

- K_5 is embedded into the torus, while $K_{3,3}$ is embedded in the Band of Moebius.
- Torus can be regarded as a sphere with a handle, so in the general case we have a sphere with multiple handles.
- The number of handles becomes a **genus**.
- A surface has **genus g**, if it is homomorphic to a sphere with g handles.
- The sphere has g=0, while torus has g=1.
- A graph that can be embedded on a surface of genus g but not on a surface of genus $g-1$, is called a **graph of genus** $\boldsymbol{g}$.
- **Theorem 11:**

  If G is a connected graph then it holds:

$$n + r = m + 2 - 2g$$

Planarity



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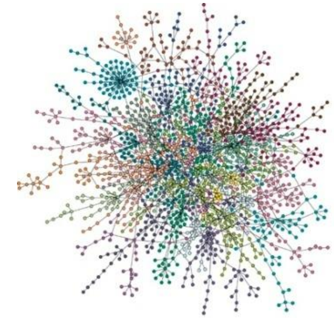
- **Theorem 12:**

  The genus $g(G)$ is not greater that the cross number $cr(G)$of a graph G:

$$g(G) \leq \mathrm{cr}(G)$$

PLANARITY



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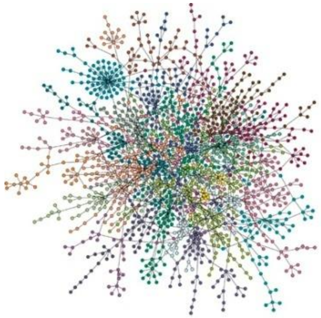
- **Corollary:**

The genus $g(G)$ of a graph G $n \geq 4$ satisfies the relation:

$$g(G) \geq \left\lceil \frac{m-3n}{6} + 1 \right\rceil$$

PLANARITY



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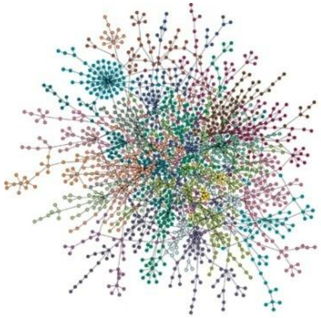
- **Theorem 13:**

The genus $g(G)$ of a complete graph K_n $n \geq 4$ satisfies the relation:

$$g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

Planarity



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- **Corollary:**

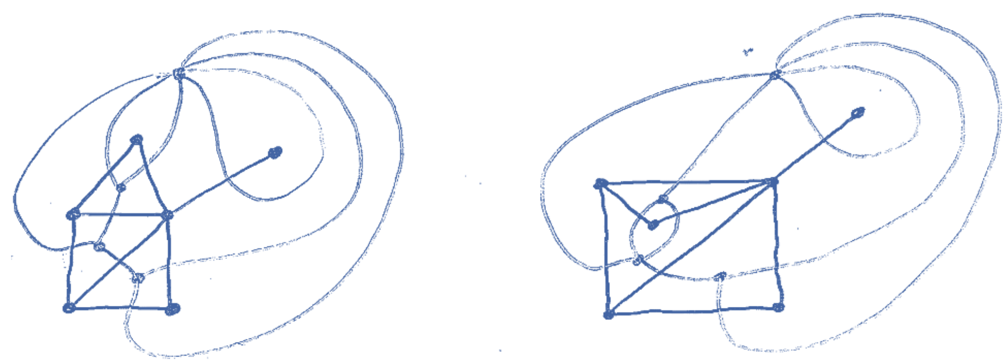
The genus $g(G)$ of a complete bipartite graph K_{n_1,n_2} satisfies the relation:

$$g(K_{n_1,n_2}) = \left\lceil \frac{(n_1-2)(n_2-42}{4} \right\rceil$$

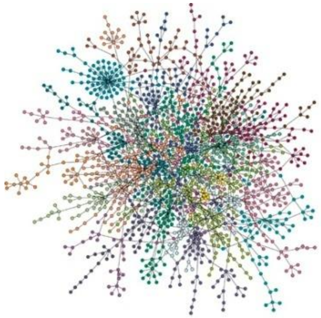
Planarity

Duality

- **Geometric Dual**:
  - In each region of G a vertex of G^* is inserted.
  - Two vertices of G^* are joined by one edge for each common edge of the corresponding regions of G.
  - For each bridge of G it is inserted at G^* a loop at the top corresponding to the regions surrounding the bridge.
  - Every edge of G^* intersects with only one edge of G.



Planarity



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- **Combinatorial Dual**:
  - A graph $\tilde{G}$ is called combinatorial dual (or, abstract dual) of a graph G if and only if there exists unambiguous match between their edges, such that the edges of a cycle of $\tilde{G}$ correspond to a vertex cut set of G.

- **Theorem 14:**

  Every plane graph G has a corresponding plane combinatorial dual graph G^*.

PLANARITY

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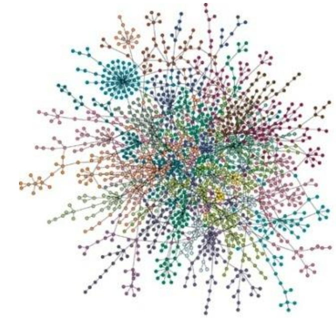
- **Corollary:**

If the graph G has a has a geometric dual graph G^*, then it holds:

$$(G^*)^* = G$$

PLANARITY



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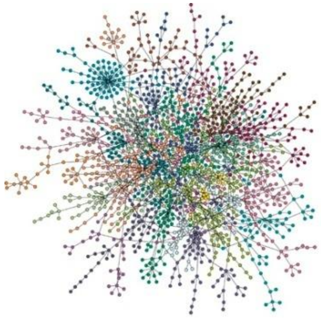
  - **Theorem 15:**

    A graph G is plane if and only if it has a combinatorial dual graph.

Planarity



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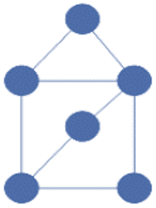
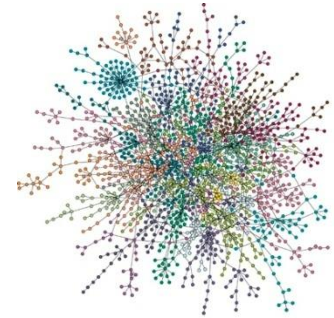
  - **Self Dual**:

A graph homomorphic to its dual is called self-dual.



Planarity



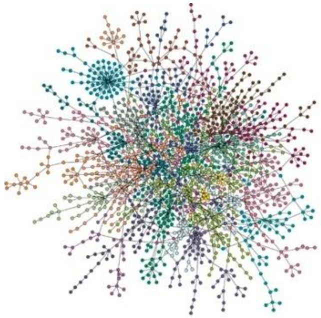
Other Planarity Criteria

- Except for the Euler theorem and the Kuratowski theorem there are two other criteria regarding the planarity of a graph.
- A **Complete Set of Basic Circuits** is a set of circles where:
  - Each circle of the graph can be expressed as a ring sum of some or all of the cycles of the set S, and
  - No circle of the set S can be expressed as a ring sum of other cycles inside S.
- **Theorem 16 (MacLane 1937):**

  A graph G is plane if only if there is a Complete Set of Basic Circuits S, such that no edges of the graph G appear in more than two cycles of S.

- The three theorems (Euler, Kuratowski, McLane) do not give effective algorithms either plane representations

Planarity



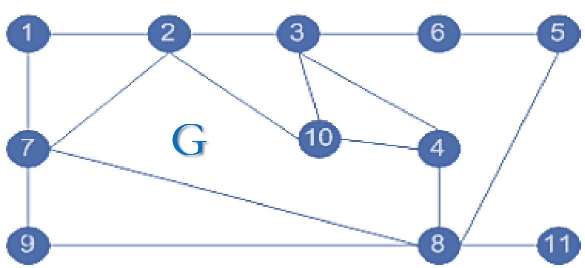
Other Planarity Criteria

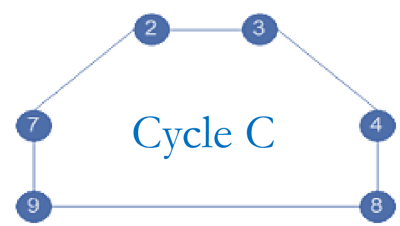
- Let graph $G(V,E)$ and subgraph $G_1(V_1,E_1) \subseteq G$.
- A piece, P, of $G(V,E)$ is called **relative** to subgraph $G_1(V_1,E_1)$ if:
  - either, an edge $e = (u,v) \in E$, where $e \notin \mathrm{E}_1$, and $u,v \in V_1$
  - or, a connected component of graph $G - G_1$ plus any edges incident on the vertices of the component
- The edges of P that belong also in G_1 are called **contact vertices**.
- If a piece has two or more contact vertices is called **segment**, or, **bridge**.
- If C is a cycle of graph G, then the embed of C partitions the plane into two regions, one inner and one outer.
- Two segments are called **incompatible** if at least two of their edges are crossed when placed in the same region of the plane defined by cycle C.

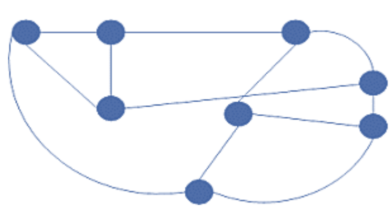
Planarity

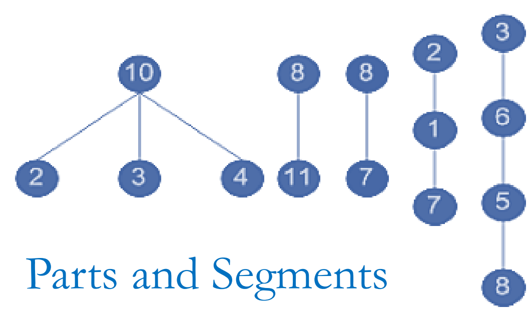
Other Planarity Criteria

- The **auxiliary** graph has vertices corresponding to incompatible segments and edges joining the vertices if the segments are incompatible.
  - The embed of pieces that are not segments is easy because they only have one point of contact with graph.
  - For the embed of segments it is constructed an auxiliary graph $P(C)$
  - This graph has as many vertices as the segments of the graph that are relative to subgraph G_1 and edges joining the vertices if parts are incompatible.





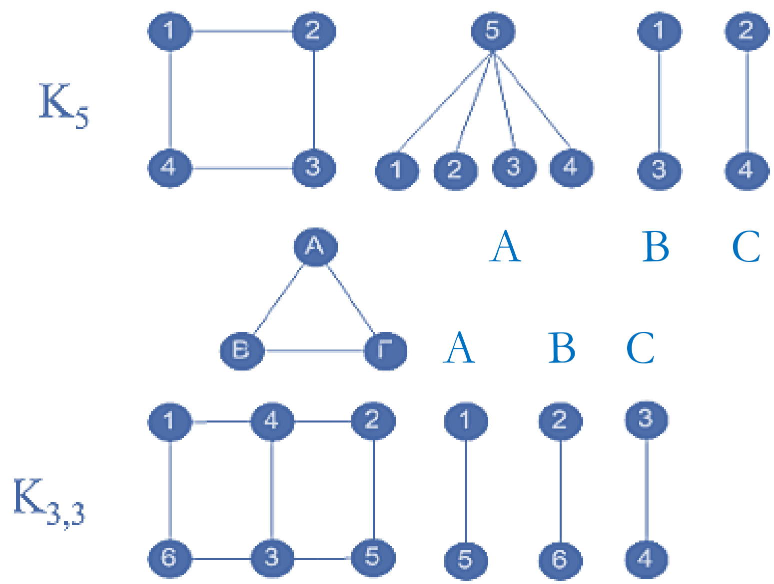

Incompatible


Parts and Segments

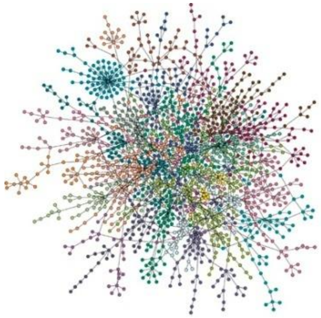
Planarity

- **Other Planarity Criteria**
  - **Theorem 17 (MacLane 1937):**

    A graph is plane iff for every circle C of G, the auxiliary graph $P(C)$ is bipartite.



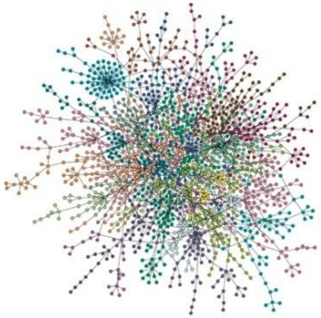
PLANARITY



- **Planarity Detection Algorithm (DMP)**
  - **Algorithm of Demoucron, Malgrange, Peruiset 1964 (DMP)**
  - **Pre-processing:**

1. If $n < 5, m < 9$, then the graph is plane
2. If $m > 3n - 6$, then the graph is non-plane
3. consider connected graphs
4. consider 2 $-$connected graphs (blocks)
5. consider simple graphs
6. produce uniform graphs without vertices of degree 2

Planarity

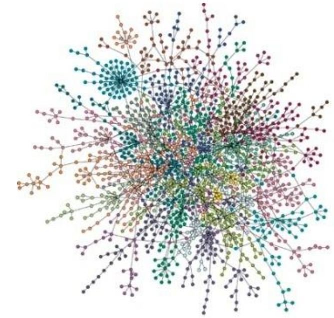


Planarity Detection Algorithm (DMP)

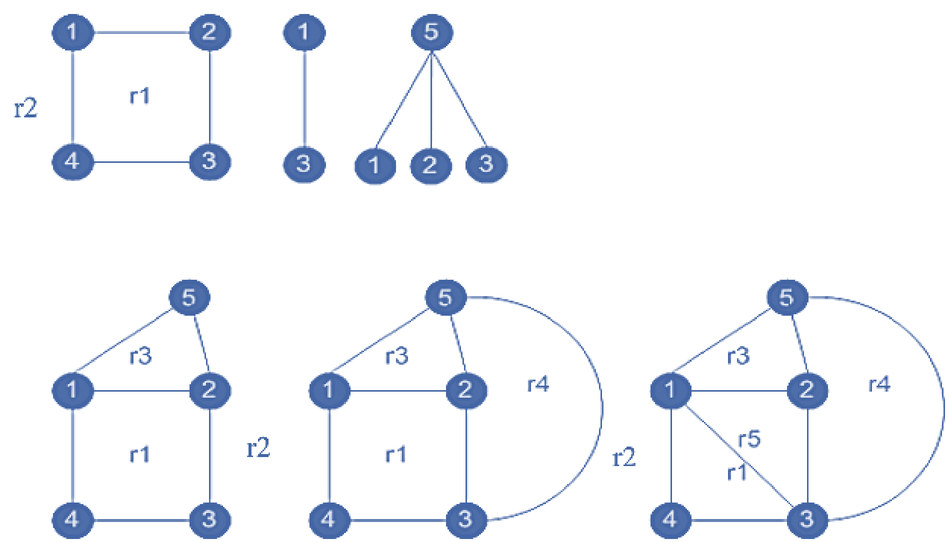
- **Algorithm of Democron, Malgrange, Peruiset 1964 (DMP)**
- **Pre-processing:**

1. If $n < 5, m < 9$, then the graph is plane
2. If $m > 3n - 6$, then the graph is non-plane
3. consider connected graphs
   - If the graph is not connected then test each component seperately
4. consider 2 −connected graphs (blocks)
   - If the graph has a cut-vertex it is adequate to check if the two blocks are plane.
5. consider simple graphs
   - If there exist loops or parallel edges, ignore them
6. produce uniform graphs without vertices of degree 2
   - contract to homomorphic graph with smaller number of vertices

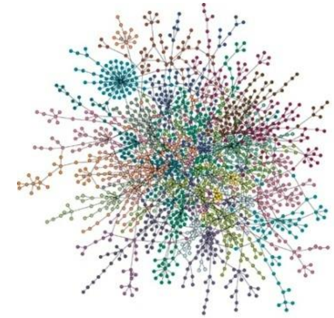
PLANARITY



- **Planarity Detection Algorithm (DMP)**
  - **Algorithm of Demoucron, Malgrange, Peruiset 1964 (DMP)**
  - **Strategy of DMP:** find a sequence embeddible subgraphs (progressively larger), starting from a circle and adding segments.
    - Starting from the cycle, segments are created.
    - For each segment we find the number of regions that it can be embedded.
    - If a segment is embedded into only one region, then it has priority.
    - In the case of a tie, we choose at random.
    - The process is repeated at most $m - n + 1$ times.



Planarity



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  - In the case of a tie, we choose at random.
  - The process is repeated at most $m - n + 1$ times.
- The DMP algorithm has complexity $O(n^4)$, however, there is also the Hopcroft-Tarjan (1974) algorithm with $O(n)$ complexity which is based on DFS, but is complex.